J. Fluid Mech. (1977), *vol.* 83, *part* 1, *pp.* 191–206
Printed in Great Britain

Variable-viscosity flows in channels with high heat generation

By J. R. A. PEARSON
Department of Chemical Engineering and Chemical Technology,
Imperial College, London

(Received 10 March 1977)

This paper presents a similarity solution for plane channel flow of a very viscous fluid, whose viscosity is exponentially dependent upon temperature, when heat generation is very large. A dimensionless formulation of the problem involves two length scales (the depth h and length l, respectively, of the channel), one velocity scale (the mean velocity V of the fluid along the channel), the thermal conductivity k, thermal diffusivity κ and viscosity μ of the fluid, and the temperature coefficient b of the viscosity. From these, two important dimensionless groups arise, the Graetz number ($Gz = Vh^2/\kappa l$) and the Nahme–Griffith number ($G = \mu V^2 b/k$). In the case of steady flow with $G^{-1} \ll Gz^{-1} \ll 1$ a thin thermal boundary layer of thickness proportional to $Gz^{-\frac{1}{3}}$ arises at each wall with an even thinner shear layer, detached from the wall and embedded in the thermal boundary layer, of thickness proportional to $Gz^{-\frac{1}{3}}(\ln G)^{-1}$, coinciding with the region of maximum temperature $(\ln G)/b$. The similarity variable is $(Pe^{\frac{1}{3}}y/x^{\frac{1}{3}})$ where Pe is the Péclet number (Vh/κ) and y and x are measured away from and along (either) boundary wall. The analogous unsteady uniform flow solution is also given.

1. Introduction

The prediction of velocity and temperature profiles for channel flows of high viscosity materials has been discussed by several authors. A recent review article has been published by Winter (1977) giving an exhaustive bibliography; although the title suggests that polymers are being considered, the analysis given effectively refers to shear and temperature-dependent viscous fluids.

The situation commonly met in confined polymer flows is such that the energy equation is dominated by three terms:

(1) heat generation due to viscous dissipation,
(2) heat conduction across streamlines, and
(3) slow changes in heat convection along streamlines.

Temperature gradients are small along streamlines and large across them. This arises when the Péclet number ($Pe = VL/\kappa$, where V is a characteristic velocity and L a characteristic length, κ being the thermal diffusivity) is large, as is usually the case in polymer processing situations. The relative importance of effects (2) and (3) above obviously introduces a ratio of length scales l/h, where l is the distance along the channel and h is the distance across the channel. High heat generation arises when $l/h \gg 1$, and again this is the situation of most significance in many processing operations. The

Graetz number ($Gz = Vh^2/\kappa l$) becomes the relevant dimensionless group comparing convection with conduction.

In many situations, there is strong coupling between the energy and momentum equations through the temperature-dependence of viscosity. Thus it is convenient to define a characteristic rheological temperature difference (b^{-1}) based on the viscosity and to compare this with various other temperature differences that arise in the flow field. One such ratio is the Nahme–Griffith number G, defined below in equation (11), which can be regarded as the ratio of the temperature rise due to heat generation and that needed to alter the viscosity by the factor e. Another, B, compares the imposed temperature difference with b^{-1}.

These ideas are discussed fully elsewhere (Pearson 1972, 1977) in the context of polymer processing. Analytical or numerical solutions for the cases where G or B are not very large have been well analysed (see Winter 1977).

A recent paper (Ockendon & Ockendon 1977) gives a very successful account of the mathematical structure of the velocity and temperature fields in the case where B is very large but G is zero, and where the relevant temperature difference is imposed by a step change in wall temperature.

We consider here the case where G is very large, and much larger than B. This covers a situation that is not irrelevant for processing situations, particularly injection moulding, for which both steady and unsteady solutions are relevant. The asymptotic solutions that we present are particularly useful in that they cover ranges of parametric variables for which numerical solutions converge slowly or not at all.

Before developing the mathematical model, it is worth considering in very simple terms the pattern of temperature development for pressure flow in a uniform channel of very viscous incompressible fluid of low thermal conductivity, whose viscosity decreases rapidly with temperature. It is simplest to consider the unsteady initial-value problem for an infinitely long channel where the walls are held at constant temperature and the flow rate is constant. Initially the temperature is uniform throughout, and the viscosity is high everywhere. For very short times the flow is adiabatic, i.e. there is no heat conduction and the temperature rises directly in proportion to the rate of dissipation of mechanical energy. For a Newtonian fluid, for example, this gives a parabolic temperature profile with the temperature rise greatest at the walls. If the thermal conductivity is low enough, this continues until the temperature rise is so large that the viscosity decreases significantly and most near the walls. Elementary stress analysis shows that the shear rate will rise preferentially near the walls and, because of the constant flow rate criterion, will fall near the centre of the channel. This in turn means that the rate of heat generation near the walls increases proportionately to that near the centre, though it decreases overall in absolute terms. Thus in the absence of any heat conduction, including no heat transfer at the walls, a plug flow will ultimately develop with increasing temperature gradients concentrated near the walls. At some stage the heat conductivity of the fluid will act to smooth out these sharpening temperature profiles in a very thin thermal boundary layer, within which an even narrower shear layer occurs, the latter concentrated close to the position of maximum temperature. In general terms we know that conduction will in the end balance generation; we can also readily see that, if any final steady-state situation arises, it will be such that the temperature will be highest at the centre of the channel and so we shall have a hot core flow rather than a cold plug flow. What we investigate is the early stage of this development from adiabatically induced plug flow to fully developed hot core flow, and suggest, plausibly even if we cannot prove rigorously, that it can be described by a relatively simple similarity solution. The same arguments are then applied to the steady-state entry flow problem to derive analogous similarity solutions.

It is difficult to compare the resulting solutions with other work because they refer to precisely those extreme values that have not been considered by other authors.

2. Mathematical formulation

2.1. *Rheology*

We consider here purely viscous fluids,† and need only use a scalar simple shear viscosity μ that is shear-rate ($\dot{\gamma}$) and temperature (T) dependent, and which can be determined experimentally. Thus, formally,

$$\mu = \mu(\dot{\gamma}, T). \tag{1}$$

It is found convenient to represent this in the form

$$\mu = C_0(\dot{\gamma}^2)^{-\alpha} \exp\{b(T_0 - T)\} \tag{1a}$$

for limited ranges of T and $\dot{\gamma}$. The representation fails for very low values of $\dot{\gamma}^2$, but this is not a region of importance for cases where heat generation, and hence $\dot{\gamma}^2$, is large. The exponential dependence upon T is wholly empirical, but nevertheless indicates a variation which correctly models the actual behaviour of polymer melts, for example. The alternative representation based on a single Arrhenius activation energy, leading to an $\exp(B/T)$ dependence, though equally good in matching experimental data, is much more difficult to handle analytically and can be preferred theoretically only if one single physical process dominates viscous flow.

The value taken by α depends upon the material (polymer melt or compound) in question. A value of about 0·3 is characteristic of many polymers at high shear rates, though there are many fluids for which $\alpha \simeq 0$ is relevant. In what follows the main calculations will be carried out for $\alpha = 0$, largely because of analytical simplicity, though the consequences of taking α arbitrarily within the permissible range $0 \leqslant \alpha \leqslant \frac{1}{2}$ will be given.

2.2. *Flow boundaries*

For simplicity, we consider here two-dimensional flow in a channel of constant depth h, and length l, where $h \ll l$. Figure 1 shows the co-ordinate system, with x measured downstream, the walls being the surfaces $y = \pm\frac{1}{2}h$. The channel walls are assumed to be held at a fixed temperature. For low conductivity polymeric fluids and metal walls,

† In an earlier paper (Pearson 1967) it was argued that this approximation was relevant for isothermal flows of simple fluids (in the Noll sense) in channels of slowly varying depth. The same argument can be applied here provided temperature changes along streamlines are slow compared with the natural (viscoelastic) time scales of the fluid. Unfortunately, evidence on time scales is usually obtained under isothermal conditions; what we need is the relaxation time scale for sudden temperature changes under conditions of nearly constant simple shear, and this has been little investigated for polymeric systems. If isothermal simple shear relaxation times are taken as relevant, then a purely viscous response proves to be relevant in simple shear after a shear of about 5, which makes the viscous approximation wholly relevant in this high Pe situation.

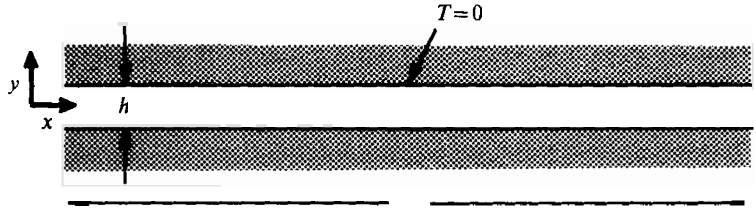

FIGURE 1. Co-ordinate system and channel dimensions.

this last is a very reasonable approximation. (Certainly an adiabatic approximation, i.e. one with no heat transfer, would be far worse. The solution for the latter extreme is rather easier to get.) It will be shown in § 5 that, where the solution for the temperature and velocity fields is of a boundary-layer character, the layer lying close to the walls, the results can be readily generalized to the case of (*a*) pipe flow, (*b*) radial flow in disk-like channels, and (*c*) slowly varying channel depth ($|\nabla h| \ll 1$) or tube radius.

2.3. *Equations of motion and energy*

To avoid unnecessary complications, we shall assume that the material is incompressible, and that inertia forces are negligible. In practice, Reynolds numbers are very small, while variations in density ρ due to temperature and pressure variations can be accommodated later. Body forces are unimportant. The velocity and temperature fields are given by

$$\mathbf{v} = \{u(x,y,t),\, v(x,y,t),\, 0\}, \quad T = T(x,y,t) \tag{2}$$

with boundary conditions

$$\mathbf{v}(x, \pm\tfrac{1}{2}h, t) = 0, \quad T(x, \pm\tfrac{1}{2}h, t) = 0. \tag{3}$$

We can define a stream function $\psi(x,y,t)$ such that

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x \tag{4}$$

satisfying the continuity equation, while the momentum equation becomes, using a lubrication approximation,

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial^2\psi}{\partial y^2}\right) \tag{5}$$

with p the pressure. The adequacy of (5) has been considered in Pearson (1967). It recognizes that $|v| \ll |u|$ and that $|\partial/\partial y| \gg |\partial/\partial x|$. It allows $\partial p/\partial x$, u and μ to be slowly varying functions of x. The variations with x will depend upon the coupling between (5) and the energy equation through μ, which is strongly temperature dependent, and so variations in T with x will have to be sufficiently slow. To maintain simplicity, we assume that the internal energy of the fluid is given by a constant specific heat c. Although this is unrealistic for polymer melts, variations in c can in principle be accommodated in a generalization of the results obtained here, most conveniently perhaps by concentrating them at a phase change, in terms of a latent heat of melting. We also suppose that heat conduction can be represented by a constant thermal conductivity k, largely for want of detailed experimental information. We thus obtain as an energy equation

$$\rho c\frac{dT}{dt} = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{6}$$

where $$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

and where only the dominant term has been retained in the viscous dissipation and heat conduction contributions. This is a quite standard approximation in the case of high Péclet number flows of viscous fluids (Nir & Acrivos 1976) and can be amply justified in all cases of practical interest.

2.4. *Dimensionless variables, parameters and equations*

It is convenient to define dimensionless variables in terms of a characteristic velocity V (related to the flow rate $Q = Vh$), the channel depth h, a characteristic viscosity μ_0, where
$$\mu = \mu_0 e^{-bT}, \tag{7}$$
and the physical parameters k, c, ρ and b already introduced.

We write
$$\xi = x/h, \quad \eta = y/h, \quad \tau = kt/\rho ch^2, \tag{8}$$
$$\phi = \psi/Vh, \quad \theta = kT/\mu_0 V^2, \quad \Pi_\xi = h^2(\partial p/\partial x)/\mu_0 V. \tag{9}$$
$$Pe = \rho c Vh/k \tag{10}$$
is the Péclet number, and very large. We also define a Graetz number
$$Gz = \rho c Vh^2/kl. \tag{10a}$$
$$G = \mu_0 V^2 b/k \tag{11}$$
is the Griffith number, and large.

Equation (5) becomes
$$\frac{\partial}{\partial \eta}\left(e^{-G\theta}\frac{\partial^2\phi}{\partial \eta^2}\right) = \Pi_\xi \tag{12}$$
with first integral, using the symmetric boundary conditions
$$\phi(\eta = \pm\tfrac{1}{2}) = \pm\tfrac{1}{2}, \tag{13}$$
$$\phi_{\eta\eta} = \Pi_\xi \eta e^{G\theta}. \tag{14}$$
The no-slip boundary condition yields
$$\phi_\eta(\eta = \pm\tfrac{1}{2}) = 0. \tag{15}$$

Equation (6) becomes
$$\theta_\tau + Pe(\phi_\eta \theta_\xi - \phi_\xi \theta_\eta) = \theta_{\eta\eta} + e^{-G\theta}(\phi_{\eta\eta})^2. \tag{16}$$
If we compress the ξ co-ordinate by writing
$$\beta = \xi/Pe \tag{17}$$
then Pe drops out of (16). It is this co-ordinate stretching that justifies the neglect of $\partial^2/\partial x^2$ terms compared with $\partial^2/\partial y^2$ terms in (6). The boundary conditions on θ are simply
$$\theta(\eta = \pm\tfrac{1}{2}) = 0. \tag{18}$$

3. Unsteady uniform flow

We consider first the case of stagnant fluid, initially all at temperature $T = 0$, suddenly set into uniform flow along the channel, considered for this problem to be effectively infinitely long. We shall take the flow rate Q to be constant. The whole solution becomes independent of x, or β, and the initial condition becomes
$$\theta(\tau = 0) = 0. \tag{19}$$

The only velocity component is that in the ξ direction, so (14) can be written

$$f(G,\tau)\,\eta = e^{-G\theta}\omega_\eta, \tag{20}$$

where $\omega = \phi_\eta$, and (16) becomes

$$\theta_\tau = \theta_{\eta\eta} + e^{G\theta} f^2 \eta^2. \tag{21}$$

Boundary conditions (13), (15) and (18) apply for all $\tau > 0$.

3.1. *Small time expansion*

For sufficiently small τ, $G\theta \ll 1$, even though $G \gg 1$. We write

$$\theta = \theta_1 \tau + \theta_2 \tau^2 + \dots, \tag{22a}$$

$$\omega = \omega_0 + \omega_1 \tau + \omega_2 \tau^2 + \dots, \tag{22b}$$

$$-f = f_0 + f_1 \tau + f_2 \tau^2 + \dots, \tag{22c}$$

where θ_i and ω_i are taken to be functions of η, and the f_i are constants. Substitution into (20) and (21) using (13) and (15), (19) being automatically satisfied, yields for the first two terms

$$\theta_1 = 144\eta^2, \quad \theta_2 = (12)^2 + (12)^4 G(\tfrac{1}{2}\eta^4 - \tfrac{1}{8}\eta^2),$$
$$\omega_0 = \tfrac{3}{2}(1 - 4\eta^2), \quad \omega_1 = -\tfrac{9}{5}(3 - 72\eta^2 + 240\eta^4)\,G,$$
$$f_0 = 12, \quad f_1 = -\tfrac{1}{5}(12)^3 G,$$

where, however, boundary condition (18) has not been satisfied. Calculation of further terms shows that the expansion cannot be expected to be valid for τ other than of order G^{-1}, and that θ, as calculated without satisfying (18), has its largest numerical value at $\eta = \pm\tfrac{1}{2}$. Inevitably thin thermal boundary layers will develop near the walls, for which the relevant similarity variable is

$$\zeta = (\eta \pm \tfrac{1}{2})/\tau^{\frac{1}{2}}. \tag{23}$$

In the neighbourhood of $\eta = -\tfrac{1}{2}$, we now add to θ given by (22a) an expansion

$$\theta^* = \tau m_1(\zeta) + \dots + \tau^n m_n(\zeta) + \dots, \tag{24}$$

where m_1, for example, obeys the equation

$$m_{1\zeta\zeta} + \tfrac{1}{2}\zeta m_{1\zeta} - m_1 = 0 \tag{25}$$

with $m_1(0) = -\theta_1(-\tfrac{1}{2}) = -36$, $m_1 \to 0$ as $\zeta \to \infty$. The solution is

$$m_1 = -(12)^2\, \mathrm{i}^2 \operatorname{erfc}(\tfrac{1}{2}\zeta) \tag{26}$$

(Abramowitz & Stegun 1964, p. 299).

Higher-order terms m_n obey similar but inhomogeneous equations with complementary functions of the form $\mathrm{i}^{2n}\operatorname{erfc}(\zeta)$.

The full details of this pair of matched expansions have not been examined. The important aspect is that a spatially maximal temperature θ of order G^{-1} is reached in times τ of order G^{-1} at a distance from the wall that is of order $G^{-\frac{1}{2}}$. Since $G\theta$ is then of order one at this layer, a pronounced effect on the viscosity and thus on the velocity and heat generation profiles will be effected. A runaway effect can be expected to take place whereby all shear and heat generation will become concentrated near the position of maximum temperature once $e^{G\theta}$ becomes large compared with one. As we shall see later, it is useful to estimate the time at which a maximum temperature $G^{-1}\ln G$ is generated, for this will lead to a viscosity variation of order $G \gg 1$. This is done in the appendix.

3.2. *Medium time expansion*

We now look for a similarity solution of (20) and (21) in which

$$\theta = g(\zeta), \quad \omega = s(\zeta) \tag{27}$$

subject to boundary conditions

$$g(0) = g(\infty) = 0; \quad s(0) = 0, \quad s(\infty) = 1. \qquad (28\,a\text{–}d)$$

This will only be relevant provided G is sufficiently large.† We want $\max g \gg G^{-1}$ so that $\max e^{Gg} \gg 1$; we also want

$$G^{-1} \ll \tau \ll 1 \tag{29}$$

so that $\zeta \gg 1$ for $\eta \ll 1$ and yet sufficient time will have elapsed for a similarity solution to develop out of the small time solution given by (22a) and (24). Only when we have obtained the similarity solution shall we be able to discuss in detail the appropriateness of boundary conditions (28).

Equation (21) becomes

$$\tau^{-1}\{\tfrac{1}{2}\zeta g_\zeta + g_{\zeta\zeta}\} + \tfrac{1}{4}e^{Gg}f^2 = 0 \tag{30}$$

in the neighbourhood of $\eta = -\tfrac{1}{2}$. Clearly

$$f = -2F(G)\tau^{-\frac{1}{2}} \tag{31}$$

for some F. From (20), using (27) and (28d), we obtain

$$1 = \int_0^\infty s_\zeta\, d\zeta = F(G)\int_0^\infty e^{Gg}\, d\zeta \tag{32}$$

and by substitution into (30)

$$g_{\zeta\zeta} + \tfrac{1}{2}\zeta g_\zeta + e^{Gg}\Big/\left(\int_0^\infty e^{Gg}\, d\zeta\right)^2 = 0. \tag{33}$$

We look for the non-trivial solution of (33) subject to boundary conditions (28a,b). (The trivial solution $g = F \equiv 0$ is in any case eliminated if the infinite limit of integration is replaced by a very large number.)

We now allow ourselves to be guided by the hope (or expectation) that g will have a sharp maximum at a distance of order one (measured in units of ζ) from the wall. Balancing the various terms in (33) leads us to write

$$Gg = \ln G - \chi(\tilde{\omega}), \quad \tilde{\omega} = \zeta - \delta \qquad (34), (35)$$

with

$$\chi(0) = \chi_0, \quad \chi_{\tilde{\omega}}(0) = 0, \tag{36}$$

$\chi_0 = o(\ln G)$, $\delta = O(1)$. χ will obey the equation

$$\chi_{\tilde{\omega}\tilde{\omega}} + \tfrac{1}{2}(\tilde{\omega} + \delta)\,\chi_{\tilde{\omega}} = e^{-\chi}\Big/\left(\int_{-\delta}^\infty e^{-\chi}\, d\tilde{\omega}\right)^2 \tag{37}$$

with boundary conditions

$$\chi(\infty) = \chi(-\delta) = \ln G. \tag{38}$$

The significance of the sharpness of the minimum of χ becomes evident in that the right-hand side of (37) will be large for very small values of $\tilde{\omega}$ and small for large values of $\tilde{\omega}$. Indeed, we can see at once by solving the linear homogeneous part of (37) that

$$\chi \sim \ln G \operatorname{erf}(\tfrac{1}{2}\zeta) \tag{39}$$

† Otherwise the solution $\omega = -F\zeta^2$ for $\zeta \to \infty$ would be significant.

for ζ or $\tilde{\omega} > 1$. Moreover, for $\ln G \gg \tilde{\omega} \ln G \gg 1$

$$\chi \sim \pi^{-\frac{1}{2}} \ln G\{e^{-\frac{1}{4}\delta^2}\tilde{\omega} + \pi^{\frac{1}{2}} \operatorname{erf}(\tfrac{1}{2}\delta)\}. \tag{40}$$

Similarly, for $\delta \ln G \gg \zeta \ln G > 0$,

$$\chi \sim \ln G\{1 - E \operatorname{erf}(\tfrac{1}{2}\zeta)\} \tag{41}$$

with E as yet unknown, and for small values of $-\tilde{\omega}$

$$\chi \sim \pi^{-\frac{1}{2}} \ln G\{\pi^{\frac{1}{2}} - E(\pi^{\frac{1}{2}} \operatorname{erf} \tfrac{1}{2}\delta - e^{-\frac{1}{4}\delta^2}\tilde{\omega})\}. \tag{42}$$

We must now obtain a solution for (37) in the region $\tilde{\omega} \leqslant O(1/\ln G)$ to match to the asymptotic forms (40) and (42). We write

$$\lambda^2 = \exp(-\chi_0) \bigg/ \left(\int_0^\infty e^{-\chi}\, d\zeta\right)^2 \gg 1, \tag{43}$$

$$\iota = \chi - \chi_0 \tag{44}$$

and use the stretched variable

$$\sigma = \lambda \tilde{\omega} \tag{45}$$

to obtain

$$\iota_{\sigma\sigma} + \frac{1}{2}\left(\frac{\delta}{\lambda} + \frac{\sigma}{\lambda^2}\right)\iota_\sigma = e^{-\iota}$$

with

$$\iota(0) = \iota_\sigma(0) = 0. \tag{46}$$

If we assume that $\lambda = O(\ln G)$, the lowest-order contribution to ι is the solution of

$$\iota_{\sigma\sigma} = e^{-\iota}. \tag{47}$$

This can readily be seen to be $\iota = -2\ln\{2e^{\sigma/\sqrt{2}}/(1 + e^{\sqrt{2}\sigma})\}$, which for large $|\sigma|$ is given by

$$\iota \sim \sqrt{2}\,|\sigma|. \tag{48}$$

On matching (48) and (40) we get

$$\lambda = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{4}\delta^2} \ln G. \tag{49}$$

Equation (43) now yields, to the same order of approximation,

$$1 = \exp(\tfrac{1}{2}\chi_0) \bigg/ \int_{-\infty}^{\infty} e^{-\iota}\, d\sigma, \tag{50}$$

whence

$$\chi_0 \sim 3 \ln 2. \tag{51}$$

Matching (48) to (42) gives

$$E = 1. \tag{52}$$

Finally, requiring the constant parts of (40) and (42) to be equal gives

$$\operatorname{erf} \tfrac{1}{2}\delta = \tfrac{1}{2}, \quad \text{i.e. } \delta = 0{\cdot}955. \tag{53}$$

It is reasonably straightforward to verify that all terms neglected are of lower order in the region of overlap. What we have shown is that, for $\ln G$ asymptotically large, a similarity solution exists with the following features:

for $\zeta = (\eta + \frac{1}{2})\tau^{-\frac{1}{2}}$ near to 0·955, θ has a maximum close to $G^{-1}(\ln G - 3\ln 2)$ which tends to

$$G^{-1} \ln G \operatorname{erfc}(\tfrac{1}{2}\zeta) \quad \text{for} \quad \zeta > 0{\cdot}955 \quad \text{and}$$

$$G^{-1} \ln G \operatorname{erf}(\tfrac{1}{2}\zeta) \quad \text{for} \quad \zeta < 0{\cdot}955.$$

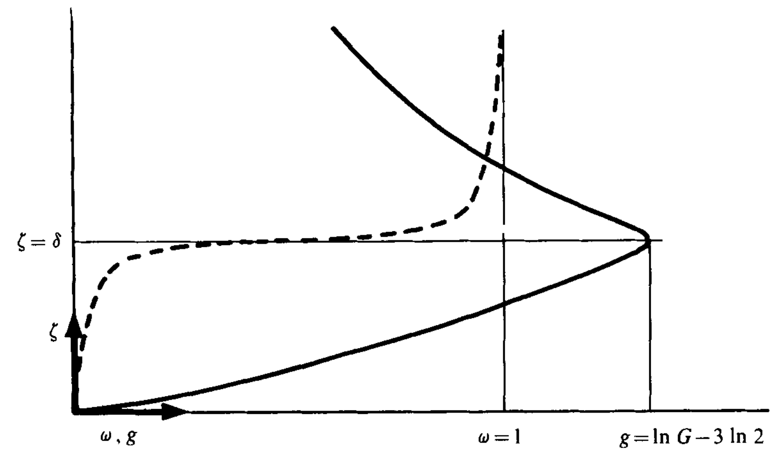

FIGURE 2. Dimensionless temperature and velocity profiles for unsteady uniform flow. ——, $g(\zeta)$, dimensionless temperature profile; – – –, $\omega(\zeta)$, dimensionless velocity profile.

It is worth observing that almost all of the volume flow Q, to the order considered, takes place as a uniform flow (a plug flow) with velocity almost V across almost all of the width h of the channel. All of the shear takes place in a layer whose thickness is of order $(\ln G)^{-1}\tau^{\frac{1}{2}}$ at a distance of order $\tau^{\frac{1}{2}}$ from the wall. Clearly we consider $\tau^{\frac{1}{2}}$ to be of order $(\ln G)^{-1}$ at most so that the asymptotic similarity solution dominates, but this can still be very large compared with the value $O(G^{-\frac{1}{2}})$ at which we assumed such a layer would form (see appendix). Figure 2 shows diagrammatically profiles for velocity and temperature.

The actual shape of the velocity profile is given by

$$s(\sigma) = e^{\sqrt{2\sigma}}/(1+e^{\sqrt{2\sigma}}).$$

We can now justify the use of the boundary conditions $g(\infty) = 0$, $s(\infty) = 1$ in (28) by noting that, even if these had been altered by terms of order G^{-1} and $(\ln G)^{-1}$ respectively, the solution obtained would still have been accurate to terms of order $(\ln G)^{-1}$ smaller than those retained.

3.3. *Longer time solutions*

It should be possible to extend the solution given in § 3.2 above to take account of the constriction of the core flow as $\tau^{\frac{1}{2}}$ became significant compared with unity. Clearly the outer boundary conditions in (28) would have to be replaced by ones taking account of the interaction of the two thermal boundary layers and of the change in width, and hence velocity, of the core flow. This is carried out to a certain extent in Ockendon & Ockendon (1977).

However, what is easier to investigate is the fully developed hot jet flow that is relevant for long times, obeying (20) and the steady energy equation

$$\theta_{\eta\eta} + e^{G\theta} f^2 \eta^2 = 0. \quad (54)$$

We anticipate that θ will have a maximum at $\eta = 0$ and so put

$$G\theta = \Phi - \phi(\eta) \quad \text{with} \quad \phi(0) = \phi_\eta(0) = 0.$$

If we write

$$\alpha = (Gf^2 e^{\Phi})^{\frac{1}{4}}\eta = A\eta \quad (55)$$

then
$$\phi_{\alpha\alpha} = \alpha^2 e^{-\phi} \tag{56}$$
with
$$\phi(\pm\tfrac{1}{2}A) = \Phi.$$

Provided A and Φ are large enough then $-\frac{1}{2}A < \alpha < \frac{1}{2}A$, $\phi_\alpha = \pm 2\Phi A^{-1}$, because $\phi_{\alpha\alpha} \sim 0$, for almost all of the range $(0, A)$ in α. The total flux will be given by

$$1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \omega(\eta)\, d\eta = 2f \int_{-\frac{1}{2}}^{0} d\rho \int_{-\frac{1}{2}}^{\rho} \eta\, e^{G\theta}\, d\eta. \tag{57}$$

On inverting the order of integration using (54), and putting $\theta_\eta(-\frac{1}{2}) = \Phi/2G$, we obtain simply

$$\Phi = \tfrac{1}{4}Gf. \tag{58}$$

It then follows, provided $2\Phi A^{-1}$ is $O(1)$, which can be checked *a posteriori*, that

$$\Phi = \ln G + 2\ln(\ln G) + \text{h.o.t.} \tag{59}$$

The difference in maximum temperature θ from that given in § 3.2 is therefore $2G^{-1}\ln(\ln G)$.

4. Steady developing flow

We consider here steady flow in the channel which is governed by equation (12), boundary conditions (13), the energy equation

$$\phi_\eta \theta_\beta - \phi_\beta \theta_\eta = \theta_{\eta\eta} + e^{-G\theta}(\phi_{\eta\eta})^2 \tag{60}$$

that is obtained from (16) and (17), the boundary condition (18) and the initial condition

$$\theta(\beta = 0) = 0. \tag{61}$$

There will be an analogy with the small time expansions given in § 3.1, relevant for very small β,† although the point (0, 0) will be special in that the term T_{xx} will in practice be important locally and so the original equation (6) would not be valid there. We shall not examine this region here. We note that the fully developed solution discussed in § 3.3 will be relevant in the limit $\beta \to \infty$.

Convection is here represented by two terms, the second of which is neglected by Winter (1977) in his more general treatment. It is included in the similarity solution given in § 4.1 below.

4.1. *Similarity solution*

We now consider a similarity solution that will be relevant for $1 \gg \beta > 0$ in terms of a similarity variable

$$\gamma = (\eta + \tfrac{1}{2})/\beta^{\frac{1}{3}} \tag{62}$$

chosen to describe boundary-layer behaviour near $\eta = -\frac{1}{2}$. We suppose that

$$\phi = \beta^{\frac{1}{3}} j(\gamma), \quad \theta = g(\gamma), \quad \Pi_\xi = 2K(G)\beta^{-\frac{1}{3}}, \tag{63}$$

where
$$j(0) = j_\gamma(0) = 0, \quad j_\gamma(\infty) = 1, \tag{64a, b}$$
$$g(0) = 0, \qquad g(\infty) = 0. \tag{64c, d}$$

† The equivalent time is that given by $t = l/V$ in (8).

On substituting into (12) and (60) we get

$$j_{\gamma\gamma} = -K\,e^{Gg},\dagger \tag{65}$$

$$g_{\gamma\gamma} + \tfrac{1}{2}g_\gamma j + K^2 e^{Gg} = 0. \tag{66}$$

From (64*c*) we have
$$1 = \int_0^\infty j_{\gamma\gamma}\,d\gamma = K\int_0^\infty e^{Gg}\,d\gamma \tag{67}$$

and again, as in (34), we write

$$Gg = \ln G - \chi(\nu), \quad \nu = \gamma - \epsilon \tag{68}$$

with
$$\chi(0) = \chi_0 = o(\ln G), \quad \chi_\nu(0) = 0, \quad \epsilon = O(1).$$

χ obeys the equation
$$\chi_{\nu\nu} + \tfrac{1}{2}j\chi_\nu = e^{-\chi}\Big/\left(\int_{-\epsilon}^\infty e^{-\chi}\,d\nu\right)^2. \tag{69}$$

Equation (69) differs from equation (37) only by the presence of j instead of $\tilde{\omega}+\delta$. The boundary conditions are still

$$\chi(\infty) = \chi(-\epsilon) = \ln G. \tag{70}$$

We surmise that the only contribution to the right-hand side of (65) will be in a very thin region near $\nu = 0$. For large enough ν, $j \to \nu - \gamma_0 + \epsilon$,

where
$$\gamma_0 = \lim_{\gamma\to\infty}(\gamma - j) = \int_0^\infty (1 - j_\gamma)\,d\gamma$$
$$= \int_0^\infty \left(1 - GK\int_0^\rho e^{-\chi}\,d\gamma\right)d\rho,$$

and so, by solving the homogeneous part of (69),

$$\chi \sim \ln G\,\mathrm{erf}\{\tfrac{1}{2}(\nu + \epsilon - \gamma_0)\}, \tag{71}$$

a result analogous to (39). Moreover, for sufficiently small ν, (71) leads to

$$\chi \sim \pi^{-\frac{1}{2}}\ln G(e^{-\frac{1}{4}(\epsilon-\gamma_0)^2}\nu + \pi^{\frac{1}{2}}\mathrm{erf}\{\tfrac{1}{2}(\epsilon-\gamma_0)\}). \tag{72}$$

For sufficiently large negative ν, $j \sim 0$ and so

$$\chi_{\nu\nu} \sim 0. \tag{73}$$

The relevant asymptotic solution, to order $\ln G$, becomes

$$\chi \sim \ln G\,|\nu|/\epsilon, \tag{74}$$

which takes the place of (42) above. As before, in the region $|\nu| \leqslant O(1/\ln G)$ we define ι and σ [(43)–(45)] to give the solution (48). On matching (48) and (72), we find that χ_0 is still given by (51) while

$$\lambda = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{4}(\epsilon-\gamma_0)^2}\ln G \tag{75}$$

instead of (49).

Finally when we match (48) to (74), and use the definition of γ_0 we find that

$$\gamma_0 = \epsilon + O(1/\ln G), \quad \epsilon \simeq \pi^{\frac{1}{2}}. \tag{76}$$

† The solution $j \sim \frac{1}{2}K\gamma^2$ for large γ is of order G^{-1} and so does not appear in an expansion based on $(\ln G)^{-1}$. It is obvious that this places a restriction on the lower limit for β for which the similarity solution will be valid since γ is actually bounded above by $\frac{1}{2}\beta^{-1}$.

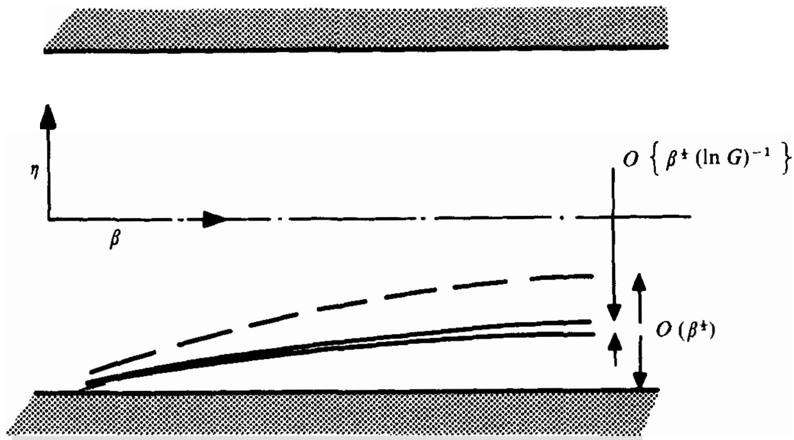

FIGURE 3. Diagrammatic indication, using partly stretched co-ordinates (η, β), of double-layer structure of temperature and flow field. The broader parabolic layer (broken curve) has a thickness $O(\beta^{\frac{1}{2}})$ and is the region in which the temperature sensibly departs from its inlet value. The thin detached layer (solid curves) has a thickness $O[\beta^{\frac{1}{2}}(\ln G)^{-1}]$ and is the region in which the velocity changes from near zero by the wall to near its core value; the temperature θ is essentially $\ln(G)/G$ in this region.

Thus we have obtained a similarity solution in which θ has a maximum close to

$$G^{-1}(\ln G - 3\ln 2) \quad \text{for} \quad \gamma = (\eta + \tfrac{1}{2})\,Pe^{\frac{1}{2}}/\xi^{\frac{1}{2}} \quad \text{close to} \quad \pi^{\frac{1}{2}},$$

tends to

$$G^{-1}\ln G\,\mathrm{erfc}\,(\tfrac{1}{2}\gamma - \tfrac{1}{2}\pi^{\frac{1}{2}}) \quad \text{for} \quad \gamma > \pi^{\frac{1}{2}}$$

and to

$$G^{-1}\ln G\,\gamma/\pi^{\frac{1}{2}} \qquad \text{for} \quad \gamma < \pi^{\frac{1}{2}}.$$

The range of $\beta = \xi/Pe$ for which the above solution is valid is similar to that for τ described in § 3.2. Assuming that the maximum for θ has to occur for $\eta + \frac{1}{2} \ll 1$, we have that the solution will be relevant for almost all of the channel length provided

$$Pe/G \ll l/h \ll Pe, \tag{77}$$

i.e. provided

$$G^{-1} \ll Gz^{-1} \ll 1. \tag{78}$$

Figure 3 illustrates the nature of the layer structure. There is no difficulty in practice in ensuring that the upper and lower limits do indeed differ by orders of magnitude.

4.2. *Pressure drop and mean temperature*

The pressure drop corresponding to the solution given by (63) is

$$\begin{aligned} P &= -\int_0^l \frac{\partial p}{\partial x}\,dx = \int_0^{l/h} \frac{2\mu_0 V}{h}\,(Pe/\xi)^{\frac{1}{2}}\,K(G)\,d\xi \\ &= \frac{8}{\sqrt{\pi}}\,Gz^{\frac{1}{2}}\,\frac{\mu_0 V l}{h^2}\,\frac{\ln G}{G}. \end{aligned} \tag{79}$$

This corresponds to a mean adiabatic temperature rise of

$$\begin{aligned} \hat{T}_{\mathrm{ad}} &= \frac{\Delta P}{\rho c} = \frac{\mu V^2}{k}\left(\frac{64}{\pi Gz}\right)^{\frac{1}{2}}\frac{\ln G}{G} \\ &= \frac{\ln G}{b}\left(\frac{64}{\pi Gz}\right)^{\frac{1}{2}}. \end{aligned} \tag{80}$$

We note that the maximum temperature given by the similarity solution is

$$T_{\max} = (\ln G)/b. \tag{81}$$

This will be greater than $\hat{T}_{\mathrm{ad}}$ provided $Gz > 20$, which is consistent with most of the range (78). The actual mean temperature of the fluid leaving the end of the channel will be

$$\hat{T}_{\mathrm{ss}} = \frac{\mu_0 V^2}{k}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial \phi}{\partial \eta}\theta\, d\eta \simeq \frac{\ln G}{b}\frac{2}{\Gamma(\frac{3}{2})\, Gz^{\frac{1}{2}}}, \tag{82}$$

where the solutions $g(\gamma)$ and $j(\gamma)$ from (63) have been used. We see that

$$\frac{\hat{T}_{\mathrm{ss}}}{\hat{T}_{\mathrm{ad}}} = \frac{\pi^{\frac{1}{2}}}{4\Gamma(\frac{3}{2})} = \tfrac{1}{2}, \tag{83}$$

which confirms what is obvious from the structure of the dissipation layer. This gives a very simple way of assessing $\hat{T}_{\mathrm{ss}}$ in the form $P/2\rho c$ for practical cases in which P is easily measured but T is not.

5. Extensions of the basic solution

5.1. *Shear-dependent viscosity*

We go back to the more general relation (1*a*) for viscosity, which in terms of the velocity field (2) becomes, instead of (7),

$$\mu = C_0 e^{-bT}(\partial u/\partial y)^{-2\alpha}. \tag{84}$$

For the non-dimensionalizing procedure employed in § 2.4, we put

$$\mu_0 = C_0(V/h)^{-2\alpha}. \tag{85}$$

For the unsteady uniform case treated in § 3 we now recover the equations

$$f_m(G,\tau) = e^{-G\theta}(\omega_\eta)^{1/m} \tag{86}$$

and

$$\theta_\tau = \theta_{\eta\eta} + (f_m \eta)^{1+m} e^{mG\theta} \tag{87}$$

instead of (20) and (21), where

$$m = (1-2\alpha)^{-1}. \tag{88}$$

If we use the similarity variable (23), then (87) and (86) become

$$\tau^{-1}(g_{\zeta\zeta} + \tfrac{1}{2}\zeta g_\zeta) + 2^{-(1+m)} e^{mGg} f_m^{1+m} = 0 \tag{89}$$

and

$$s_\zeta \tau^{-\frac{1}{2}} = -(\tfrac{1}{2}f_m)^m e^{mGg}. \tag{90}$$

It is at once clear that f_m cannot be chosen so as to make (89) and (90) simultaneously independent of τ. However, we note that the generation (viscous dissipation) term in (87) was significant only in a very thin layer whose length scale could be separately chosen, and we now suppose that in order to satisfy (89) and (90) we allow both the maximum temperature Gg and the length scale of σ to depend slowly on τ.†

Thus for ζ near to δ_m, we write

$$Gmg = C_m(G,\tau) - \iota(\sigma) \tag{91}$$

† A two-time expansion should strictly be introduced at this stage, but the intention is only to outline the nature of the modification to earlier results and so a brief and simplified, if unrigorous, outline is given here.

with
$$\zeta = \delta_m + B_m(G,\tau)\,\sigma. \tag{92}$$
From (90), using
$$\int_{-\infty}^{\infty} \frac{ds}{d\sigma}\,d\sigma = 1, \quad \int_{-\infty}^{\infty} e^{-\iota} = 2\sqrt{2}$$
we get
$$\tau^{\frac{1}{2}} B_m f_m^m e^{Gm} = 2^{m-\frac{3}{2}}. \tag{93}$$
From (89), and requiring that ι obey (47), we get
$$\tau m G\, e^{C_m} f_m^{1+m} B_m^2 = 2^{1+m}. \tag{94}$$
As in § 3.2, equations (39) and (41), we require that
$$\iota \simeq \begin{cases} A_m \operatorname{erf}(\tfrac{1}{2}\zeta) & \text{for} \quad \zeta > \delta_m, \\ D_m - E_m \operatorname{erf}(\tfrac{1}{2}\zeta) & \text{for} \quad \zeta < \delta_m, \end{cases}$$
with A_m, D_m, E_m slowly varying functions of τ. On matching we have
$$A_m = D_m = E_m = C_m, \quad \delta_m = 0{\cdot}955,$$
$$B_m = \sqrt{2\pi}\exp(-\tfrac{1}{4}\delta_m^2)/C_m$$
where
$$C_m \sim \ln(G^m \tau^{\frac{1}{2}(m-1)}), \tag{95}$$
$$f_m \sim G^{-1}\tau^{-\frac{1}{2}} C_m \tag{96}$$
to a first approximation. We note that $m > 1$ with a typical value of 3 for rubbery thermoplastics. The Newtonian solution given in § 3 involved relative errors of order $(\ln G)^{-1}$. Here we find on resubstitution that further errors of order $(|\ln \tau|)^{-1}$ are introduced. But we have already shown that
$$G^{-1} < \tau \ll 1$$
so
$$1 \ll |\ln \tau| < \ln G$$
and no basic change in the asymptotic structure is involved.

Very similar remarks apply to the steady-state entry flow problem, where we find that the maximum temperature is given, from (91) and (95), by
$$G\theta = \ln(G\beta^{\frac{1}{2}(m-1)/m})$$
and the pressure drop by (79) with $\ln G$ replaced by
$$\ln\{G^m\, Gz^{\frac{1}{2}(1-m)}\}.$$

5.2. *Pipe flow*

The approximations given in § 3 or § 4 require that the velocity gradient only be significant in a thin layer at a distance from the wall small compared with the channel depth. The argument can be extended trivially to the case of cylindrical pipes of general cross-section provided that the local radius of curvature of the pipes be large compared with the distance from the wall to the hot sheared layer. The 'equivalent' channel will have a half-width equal to area/circumference, i.e. the hydraulic mean radius. V remains the uniform core velocity.

5.3. *Flow in a channel of slowly varying depth*

We suppose that $h = h(x)$ such that $|dh/dx| \ll 1$. Without going into the details of the mixed expansions that should strictly be discussed, or equivalently of the order in which various limits are taken, we find that if we define a new variable z by

$$h\,dz = h_0\,dx, \quad z(0) = 0 \tag{97}$$

then z can take the place of x in the similarity solutions obtained in § 4. This gives the relevant pressure drops in terms of the modified channel length, provided the total relative changes in h are only of order 1, so that $\ln G$ is unambiguously defined. It must be emphasized that the changes in h must be sufficiently slow. An analogous result applies to cone-like pipes of very small semi-angles.

5.4 *Radial flow in a disk-like channel*

Here we consider flow from an entry at $r = r_0$ to an exit at $r = r_1$ in a channel whose depth $h = h(r)$ is such that

$$h_{\max} \ll r_0, \quad |dh/dr| \ll 1.$$

Locally the flow approximates the channel flow. If we define a new variable q by

$$h\,r_0^{\frac{1}{2}}\,dq = h_0\,r^{\frac{1}{2}}\,dr, \quad q(r_0) = 0 \tag{98}$$

then q takes the place of x in the similarity solution of § 4. Pe is defined in terms of its value at r_0.

Appendix

As we have seen from (23)–(26), the effect of conduction is limited to a distance of order $\tau^{\frac{1}{2}}$ from the wall and so, provided we never approach values of order unity, we can solve the purely adiabatic equation

$$\theta_\tau = e^{G\theta} f^2 \eta^2, \tag{A 1}$$

where f is a function of τ given by the flux requirement

$$\int_0^{\frac{1}{2}} d\rho \int_\rho^{\frac{1}{2}} \omega_\eta\, d\eta = f \int_0^{\frac{1}{2}} d\rho \int_\rho^{\frac{1}{2}} \eta\, e^{G\theta}\, d\eta = \tfrac{1}{2}. \tag{A 2}$$

Writing

$$f^2\, d\tau = d\tau^*, \tag{A 3}$$

we get, on integrating (A 1), using $\theta(\tau^* = 0) = 0$,

$$e^{-G\theta} = 1 - \eta^2 G\tau^*. \tag{A 4}$$

Clearly τ^* can never exceed $4G^{-1}$, for at $\eta = \frac{1}{2}$ this gives $\theta \to \infty$. From (A 2) we get

$$\begin{aligned} f &= \frac{1}{2}\left(\int_0^{\frac{1}{2}} d\rho \int_\rho^{\frac{1}{2}} \frac{\eta}{1-\eta^2 G\tau^*}\, d\eta\right)^{-1} \\ &= G\tau^* \left(\int_0^{\frac{1}{2}} \ln \frac{(1-\rho^2 G\tau^*)}{(1-\frac{1}{4}G\tau^*)}\, d\rho\right)^{-1} \\ &= G\tau^* \left(\frac{1}{(G\tau^*)^{\frac{1}{2}}} \ln \frac{1+\frac{1}{2}(G\tau^*)^{\frac{1}{2}}}{1-\frac{1}{2}(G\tau^*)^{\frac{1}{2}}} - \tfrac{1}{2}\ln(1-\tfrac{1}{4}G\tau^*)\right)^{-1}. \end{aligned} \tag{A 5}$$

When $\theta_{\max} = G^{-1} \ln G$, by using (A 4)

$$G\tau^* = 4(1 - 1/G), \tag{A 6}$$

and the relevant value of τ, using (A 3), is given by

$$\tau = \frac{1}{G}\int_0^{4(1-1/G)} \left[\frac{1}{A^{\frac{1}{2}}}\ln\left(\frac{1+\frac{1}{2}A^{\frac{1}{2}}}{1-\frac{1}{2}A^{\frac{1}{2}}}\right) - \tfrac{1}{2}\ln\left(1-\tfrac{1}{4}A\right)\right]\frac{dA}{A^2} \tag{A 7}$$

and this can be shown to be of order $1/G$. At this stage we expect the similarity solution to take over, through the effect of conductivity.

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